Transitions to amplitude death in a regular array of nonlinear oscillators

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(Received 8 February 2007; published 9 July 2007)

We investigate amplitude death in a regular array of nonlinear oscillators. We find that the system undergoes three transitions during the road to complete amplitude death. The first two are related to partial amplitude death: one induces partial amplitude death and the other enhances it. The last transition establishes complete amplitude death. We also find that the first two belong to a second-order transition and are universal while the last one is strongly dependent on the system parameters.

DOI: 10.1103/PhysRevE.76.016204 PACS number(s): 05.45.Xt, 05.40.−a

I. INTRODUCTION

Coupled nonlinear oscillators provide a simple but powerful mathematical model for simulating the collective behavior of a wide variety of systems that are of interest in physical, chemical, and biological sciences [1–4]. Among the collective behaviors, amplitude death, which refers to a situation where individual oscillators cease to oscillate when coupled and go to an equilibrium solution instead, has been actively investigated [5–9]. Generally for the occurrence of amplitude death, either a large mismatch of the oscillator frequencies [10,11] or the existence of time delay in the coupling [12–16] is required. Theoretical and numerical studies have obtained great achievements on amplitude death in systems of two oscillators or populations of oscillators globally coupled together. However, understanding of the route to amplitude death in populations of oscillators coupled locally and without time delay is not enough [7,16–18]. In Ref. [19], amplitude death on an array of oscillators with the nearest neighbors has been discussed. The authors considered a special case where the natural frequencies of the oscillators are distributed in a regular monotonic trend. Partial amplitude death was found where amplitude death occurs in regions with a relatively large gradient of natural frequencies (in contrast, the complete amplitude death is defined as the one where all oscillators are dead). Furthermore, the authors found that partial amplitude death is weakened by introducing random deviations from the linear trend of frequencies. Amplitude death for oscillators with a regular monotonic trend of natural frequencies has been discussed also on complex networks [20] where the influences of the topological properties of the network on partial amplitude death were discussed. However, these works did not consider how the system makes a transition to either complete or partial amplitude death.

In this work, we consider the transition to amplitude death in coupled oscillators defined on a regular array with nearest coupling. We assume that the natural frequencies of the oscillators are distributed in a random way. We find three stages during the transition to complete amplitude death when the coupling strength is increased from zero. In the first stage, the oscillators are divided into two groups: In one group, the oscillators coordinately decrease their amplitude to death though they may not be connected directly. The other group consists of a number of small clusters in which the oscillators have built mutual synchronization to prevent amplitude death. The first stage describes the transition to a partial amplitude death. In the second stage where most of the clusters go to amplitude death in a coordinate way, another transition happens which enhances the existing partial amplitude death. In the final stage, the number of remaining clusters decreases until complete amplitude death is established. We also find that transitions to partial amplitude death in the first two stages are second-order ones and independent of the system parameters, such as the sample of natural frequencies. In contrast, the transition to complete amplitude death in the third stage is sensitive to the system parameters.

II. MODEL AND THE NUMERICAL RESULTS

We consider N coupled nonlinear oscillators with periodic boundary condition

$$\dot{Z}_i(t) = [1 + i\omega_i - |Z_i|^2]Z_i(t) + k[Z_{i+1} + Z_{i-1} - 2Z_i] \times (i = 1,2, \ldots ,N), \qquad (1)$$

where $Z_i(t)$ is a complex number which represents the state of the ith oscillator at time t, $k \geqslant 0$ is the coupling strength, and $\omega_i \in \mathbb{R}$ are the natural frequencies of the oscillators. The frequencies are assumed to be randomly distributed according to the uniform distribution $g(\omega)=\frac{1}{\Delta\omega}$ where $\Delta\omega = \max\{|\omega_i - \omega_j|\}$ $(i,j=1,\ldots,N)$. Without coupling, each oscillator has an unstable equilibrium point at $|Z_i|=0$ and an asymptotically stable limit cycle of radius $|Z_i|=1$. Once the coupling is switched on, the dynamics of the system becomes complex. Especially for proper coupling strength k and the sample $\omega_1,\ldots,\omega_n$, the static state with $|Z_i|=0$ for all i may become stable and complete amplitude death occurs. The stability of the complete amplitude death can be analyzed through the linearization of Eq. (1) at $|Z_i|=0$:

$$\delta\dot{Z}_i(t) = (1 + i\omega_i)\delta Z_i + k(\delta Z_{i+1} + \delta Z_{i-1} - 2\delta Z_i). \qquad (2)$$

Let A be the matrix of Eq. (2); we have $\mathrm{Re}(\mathrm{Tr}\,A)=n(1-2k)$. The necessary condition for complete amplitude death can be obtained as

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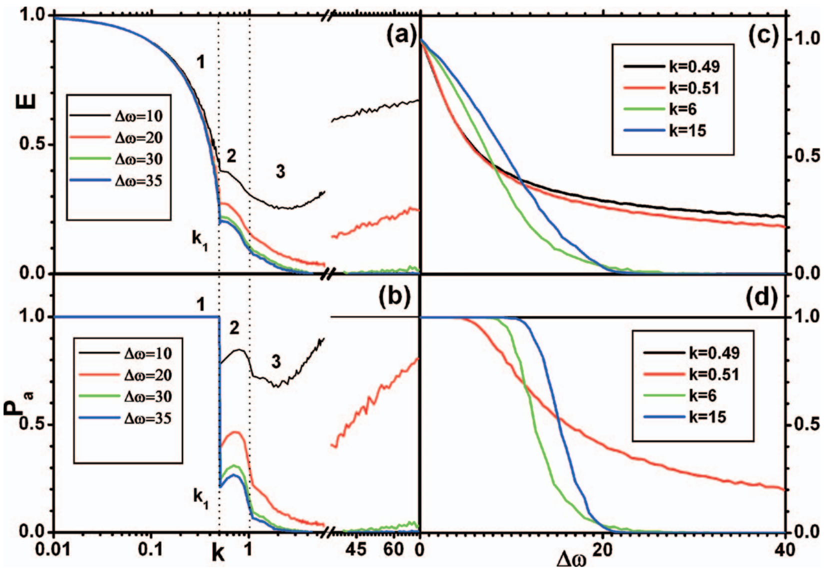

FIG. 1. (Color online) $N = 1000$. The results are averaged for 100 realizations for each $\Delta\omega$. (a) Dependence of E on the coupling strength k. Three stages are presented as indicated by 1, 2, and 3. (b) Dependence of P_a on the coupling strength k. (c) and (d) Dependence of E and P_a on $\Delta\omega$, respectively.

$$\mathrm{Re}(\mathrm{Tr}\, A) \leqslant 0 \Rightarrow k > \frac{1}{2}. \tag{3}$$

Unfortunately, the eigenvalues of the matrix A cannot be obtained rigorously. To investigate the transition of the coupled system to complete amplitude death, we resort to numerical simulations. The transition to complete amplitude death can be manifested by two quantities: the normalized mean "incoherent" energy $E = \frac{\langle \Sigma_{i=1}^{N} |Z_i|^2 \rangle}{N}$, where $\langle \cdot \rangle$ denotes averaging over time, and the percentage of active oscillators, $P_a = \frac{\langle N_a \rangle}{N}$, where N_a is the number of active oscillators with nonzero $|Z_i|$. $0 < P_a < 1$ refers to the presence of partial amplitude death and complete amplitude death occurs when both E and N_a become zero.

We first let $N = 1000$ and investigate the dependence of E and P_a on coupling strength k for different $\Delta\omega$. The results presented in Figs. 1(a) and 1(b) show that, for small $\Delta\omega$, E and P_a never reach zero and complete amplitude death is impossible. However, for sufficiently large $\Delta\omega$, complete amplitude death could exist in a range of the coupling strength. From Figs. 1(a) and 1(b), we know that there exist three stages during the transition to complete amplitude death. In the first stage where $k \lesssim 0.5$, the equity of $P_a = 1$ is held and the curves of E for different $\Delta\omega$ collapse together. Beyond $k \approx 0.5$, P_a drops sharply to a nonzero value from unity, which indicates that some oscillators become dead while the others remain active. In other words, a state of partial amplitude death is born around $k \approx 0.5$. Therefore, the first stage describes the process of transition in Eq. (1) to partial amplitude death. At the onset of partial amplitude death, the number of "dead" oscillators increases with $\Delta\omega$. In the second stage, which ranges from $k \approx 0.5$ to $k \approx 1$, the different curves of E and P_a show a large discrepancy. However, the patterns of E and P_a against k are the same for different $\Delta\omega$ provided that $\Delta\omega$ are sufficiently large. In the third stage where $k \gtrsim 1$, both E and P_a vary slowly with k. In this stage, the system of Eq. (1) for large $\Delta\omega$ will finish the transition to complete amplitude death. Generally the critical coupling strength k_c decreases with $\Delta\omega$. However, it is also important to note that the onset of complete amplitude death is strongly dependent on the sample $\omega_1, \ldots, \omega_n$, which can be proved numerically (results are not shown here). Such an observation holds true even for larger N—for example, $N = 10\,000$, which is contrary to the "all-to-all" coupling case where the amplitude death is not sensitive to the different sample $\{\omega_i\}$ for large N. Actually such a difference roots in the fact that, for large N, the arrangement of the natural frequencies $\{\omega_i\}$ can be ignored in all-to-all coupled systems while it is important in a locally coupled one. It is worth noting that the sensitivity of complete amplitude death to the different sample $\{\omega_i\}$ for large N also exists when the complete amplitude death loses its stability at larger k.

To get a general view of the transition to complete amplitude death in the system of Eq. (1), it is also necessary to investigate how E and P_a change with $\Delta\omega$. The results are presented in Figs. 1(c) and 1(d). From them, we can find that the minimum of $\Delta\omega$ required for complete amplitude death decreases with k. The information on partial amplitude death can be acquired from the curves of P_a. As revealed in Fig. 1(d), there is no partial amplitude death for small k no matter how large $\Delta\omega$ is—for example, $k = 0.49$. Figure 1(d) also states that $\Delta\omega_c$ for the occurrence of partial amplitude death increases with coupling strength k.

As mentioned above, there exists a series of transitions to the state of complete amplitude death. However, we do not have a clear picture yet of the underlying process, especially of the mechanisms behind the onset of complete and partial amplitude deaths and of the microscopic dynamics in the different stages. To gain insight into them, we first investigate how the oscillators are distributed on the complex Z plane. To make the distribution clear, we let $N = 10\,000$. The results for different k are presented in Fig. 2(a). For small k, the oscillators are distributed on one ring. With an increase of k, the radius of the ring decreases from $|Z| \approx 1$. However, when k becomes a little large—for example, $k > 0.3$—other groups of oscillators except for the first ring are formed: a second ring and some scattered set. Different groups play their roles differently at three stages: In the first stage, the first ring decreases its radius until it collapses onto $|Z| \approx 0$ at

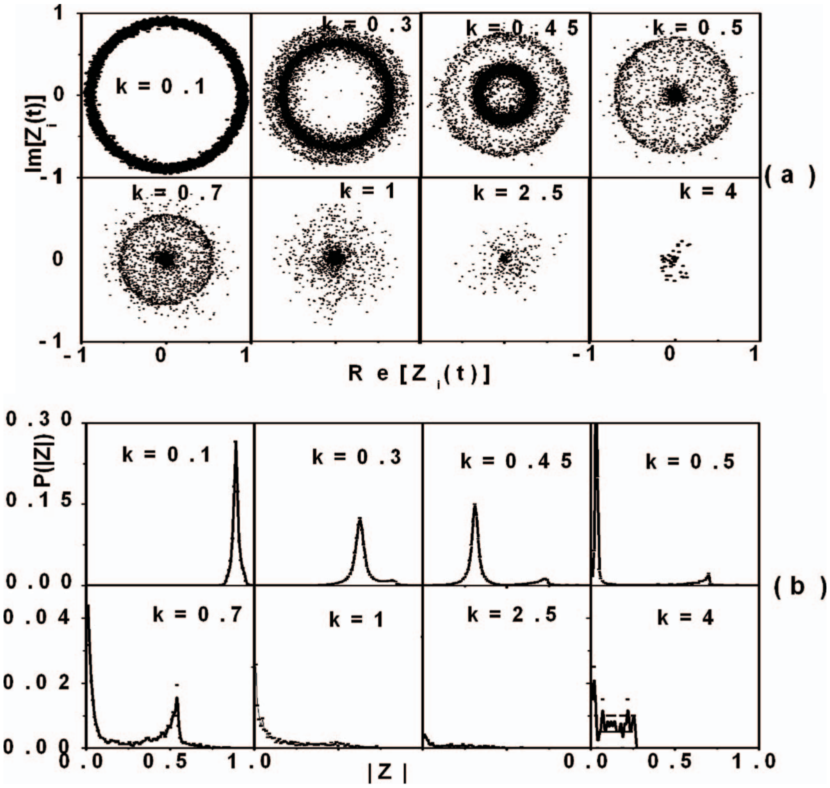

FIG. 2. $N=10000$, $\Delta\omega=30$. (a) The snapshots of all oscillators on the complex plane of Z for different coupling strengths. Qualitatively, the distribution of the oscillators on the Z plane is independent of time. (b) The distributions of the oscillators on the amplitude $|Z|$ for different coupling strengths. $P(|Z|)$ in the plot for $k=4$ is enlarged by 100.

around $k=0.5$ while the second ring only shrinks prominently in the second stage and down to zero at around $k=1$; the final transition to complete amplitude death in the last stage is finished by the collapse of the scattered set onto $|Z_i|=0$. In Fig. 2(b), the distributions of oscillators on amplitudes for different k are plotted also. The rings in Fig. 2(a) are represented by the peaks at nonzero $|Z|$. The first peak approaches $|Z|=0$ at $k\approx 0.5$ and the second one at $k\approx 1$, which signals two transitions: the first one induces partial amplitude death and the second one enhances it. Two other points can be drawn from Fig. 2(b): (i) Most of the oscillators crowd into the first ring for $\Delta\omega=30$ and the number of oscillators contained in the scattered set is the smallest. (ii) There is no dominant peak anymore for the scattered set in the third stage and the oscillators with large amplitude can persist until the coupled system reaches complete amplitude death.

The statistical information on the transition to the complete amplitude death is given in Fig. 2. However, it does not present a clear microscopic dynamics during the transition. To unfold it, it is helpful to plot the instantaneous amplitude $|Z_i(t)|$ against its position (it can be tested numerically that the fluctuation of the amplitude over time is weak). To make the plots clear, only a partial view is given in Fig. 3. During the first stage where the partial amplitude death is to be built, most of the oscillators [these oscillators belong to the first ring in Fig. 2(a)] decrease their amplitude to zero continuously. However, in this stage we also find the formation of some clusters which resist decreasing their oscillation amplitudes. Further numerical simulations reveal that those clusters are formed only in the region where the differences of the natural frequencies among neighboring oscillators are small and mutual synchronization has been established. According to the requirements for amplitude death, we know that it is the establishment of mutual synchronization within these clusters that prevents them from amplitude death and prevents the system of Eq. (1) from complete amplitude death. The synchronized clusters as a mechanism for resisting amplitude death have been discussed by Rubchinsky *et al.* [19] though they consider a special case as mentioned in the Introduction. Now it is interesting to ask why most oscillators reach amplitude death in a coordinate way as $k\to 0.5$ though they are not connected directly. A heuristic

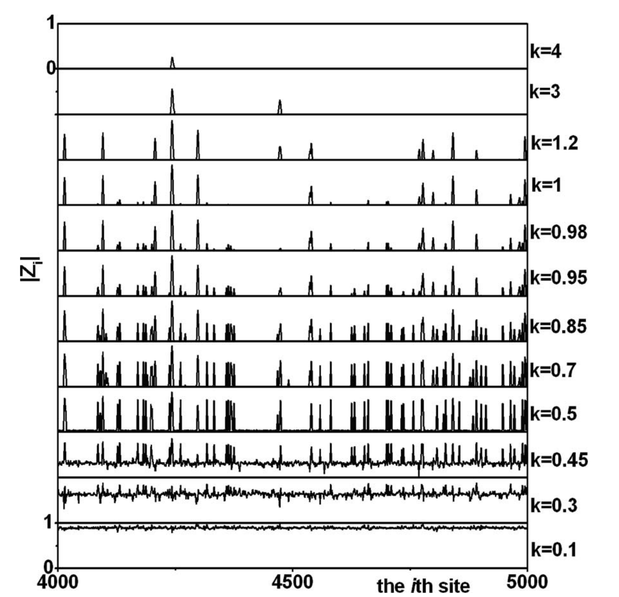

FIG. 3. The snapshots of $|Z_i|$ against the position of the oscillators for different k. $N=10\ 000$, $\Delta\omega=30$.

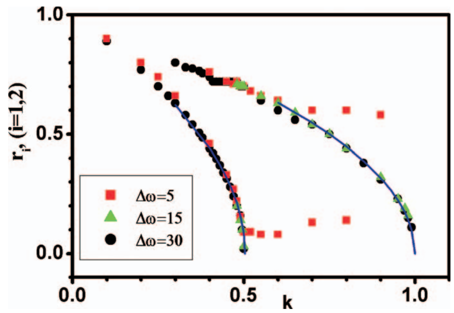

FIG. 4. (Color online) The order parameters r_i $(i=1,2)$ are plotted against the coupling strength k for different $\Delta\omega$. The fitting curves for the r_i $(i=1,2)$ follow the power law $r_i \sim (k_{c,i}-k)^{0.5}$ where $k_{c,1} \simeq 0.5$ and $k_{c,2} \simeq 1$, which are independent of $\Delta\omega$ provided that $\Delta\omega$ are not so small.

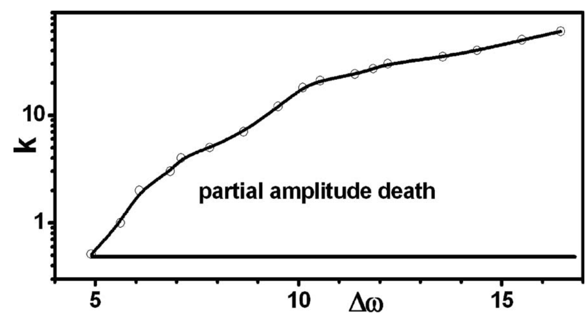

FIG. 5. The stability region of the partial amplitude death induced by the first transition. The region is bounded by the line $k \simeq 0.5$ and a monotonically increasing curve.

explanation can be given as follows. Rewrite Eq. (1) as $\dot{Z}_i(t)=[1-2k+i\omega_i-|Z_i|^2]Z_i(t)+k[Z_{i+1}+Z_{i-1}]$. Without the second term $k[Z_{i+1}+Z_{i-1}]$, it is clear that the amplitude of the ith oscillator decreases with k until the solution of $Z_i=0$ becomes stable for the ith oscillator when $k \geqslant 0.5$. On the other hand, when k approaches 0.5 from zero, most oscillators, except for the synchronized cluster, behave incoherently and the interaction term $k[Z_{i+1}+Z_{i-1}]$ plays an irrelevant role in the dynamics of the ith oscillator, which also leads to the interaction term $k[Z_{i+1}+Z_{i-1}]$ decreasing its strength with k also. As a result, oscillators not falling into the synchronized clusters become dead at $k=0.5$, the necessary condition for complete amplitude death obtained in Eq. (3). Actually the explanation holds for amplitude death in the all-to-all coupled system where the lower bound of the coupling strength k for amplitude death can be determined in this way also. In the second stage where $k \in (0.5,1)$, we can find that the highest amplitude in many clusters [the oscillators in these clusters form the second ring in Fig. 2(a)] is decreased to zero in a coordinate way once more when the coupling strength increases. Similar to the first stage, there exist some clusters (the oscillators in them form the scattered set mentioned above) which resist decreasing their amplitudes as others. In the final stage, those surviving clusters go to amplitude death independently. In this stage, the transition to complete amplitude death is characterized by a decrease of the number of the clusters. When complete amplitude death steps in depends on the properties of the last cluster. Therefore, as found in Fig. 1, the stability of complete amplitude death is strongly dependent on the sample $\{\omega_i\}$. The mechanism here is also the one for the transition occurring at larger k as shown in Fig. 1. Such a dependence can be easily demonstrated if we reconsider Eq. (1) as the one with only a small number of oscillators. In the remodeling, amplitude death in the neighborhood of the cluster is mimicked by fixed value boundary conditions $|Z_B|=0$. We will not go into the details of the amplitude death in small systems.

Up to now, we have characterized the different stages during the transition to complete amplitude death and already know that the final transition to complete amplitude death is sensitive to the system parameters and the sample of natural frequencies. It is still a problem whether the transitions to partial amplitude death in the first two stages are generic. To answer this question, we let the radius r_i $(i=1,2)$ of the first and second rings [or the positions of peaks shown in Fig. 2(b)] be the order parameters. r_1 and r_2 against k are plotted in Fig. 4. Interestingly, both r_1 and r_2 can be fitted by the law $r_i \propto (k_{c,i}-k)^{0.5}$ $(i=1,2)$, which indicates two second-order transitions. As shown in Fig. 4, the nature of the second-order transitions in these two stages and the critical coupling strengths $k_{c,1} \approx 0.5$ and $k_{c,2} \approx 1$ are not dependent on $\Delta\omega$ even if complete amplitude death is not possible for small $\Delta\omega$.

III. CONCLUSION

In discussion, we have studied the transition to complete amplitude death in coupled oscillators defined on a regular array. Three transitions are characterized: the first two are the transitions to partial amplitude death and the last is the one to complete amplitude death. We find that the transitions to partial amplitude death are independent of the system parameters and the one to complete amplitude death strongly depends on the system parameters. To be specific, the observations presented in this paper are not dependent on the periodic boundary condition; we can reproduce all of them in the system with the no-flux boundary condition. Now it is significant to make a discussion of the relation between the first transition to partial amplitude death and the one to complete amplitude death in the all-to-all coupled system. First, around the transition point, the order parameter in the system described by Eq. (1) follows a power law which is the same as the one in the all-to-all coupled system. Second, for Eq. (1), we plot the parameter regime for partial amplitude death in Fig. 5 where the regime is bounded by the necessary condition for complete amplitude death [see Eq. (3)], $k \approx 0.5$, and a monotonically increasing curve of $k(\Delta\omega)$. The stable region of partial amplitude death induced by the first transition is similar to that for complete amplitude death in the all-to-all coupled system (see Fig. 3 in Ref. [10]). All of these indicate that the first transition to partial amplitude death in Eq. (1) is a variant of the transition to complete amplitude death in the all-to-all coupled system.

ACKNOWLEDGMENTS

This work is supported by Grant No. 10405004 from CNSF and the project sponsored by SRF for ROCS, SEM

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